Boundary effects in some stochastic geometric models

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joint work with

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Largest nearest neighbour link

Given a point cloud $\mathcal{X}$ in a metric space, the LNNL tells us how far the most isolated point is from the others

- $L_n = \max_{x \in \mathcal{X}} dist(x, \mathcal{X} \setminus \{x\})$
- $L_n = \inf\{r : \deg(x) \geq 1,\ \forall x \in G(\mathcal{X}, r)\}$
- $L_n = \inf\{r : \mathcal{X}(B(x, r)) \geq 2,\ \forall x \in \mathcal{X}\}$

similarly $L_{n,k}$

Full coverage threshold

$$R_{n,k} = \inf\{r : \mathcal{X}(B(x,r)) \geq k, \ \forall x \in A\}$$

- $L_n \leq R_{n,2}$
- $L_{n,k} \leq R_{n,k+1}$

Connectivity threshold

> ⓘ A graph is k-connected, denoted $\mathcal{K}_k$, if the removal of $k-1$ vertices does not disconnect the graph.

$$M_{n,k} = \inf\{r : G(\mathcal{X}, r) \in \mathcal{K}_k\}$$

X-X X-X

$$L_{n,k} \leq M_{n,k}$$

Result

Let $\mathcal{X} = \mathcal{P}_n$. Under conditions on A, we have the strong law

$$\frac{nL_n^d}{\log n} \sim \frac{nM_n^d}{\log n} \sim \frac{nR_n^d}{\log n} \sim c$$

where c depends on the geometry of A.

- general $k = k(n)$
- binomial process
- non-uniform

- Penrose, PTRF'22, book
- Penrose, Y, Higgs, '23

Finite convex polytope

Let $k = k(n) \sim \beta \log(n)$ with $\beta \in [0, \infty]$. Then

$$c = \max_{\phi \in \Phi^*(A)} \frac{\hat{H}_\beta(D(\phi)/d)}{f_\phi \rho_\phi}$$

- $\hat{H}_0(x) = x, \hat{H}_\infty(x) = 1$ (chg scale to $k(n)$ when $\beta = \infty$)
- $\Phi^*(A)$: faces of A, including A^o viewed as d-dim face
- $D(\phi)$ is the dimension of the face ϕ
- $f_\phi = \inf_{x \in \phi} f(x)$ where f is density
- $\rho_\phi = |B(o, 1) \cap \kappa_\phi|$ angular volume of face ϕ

Proof: granulation

Want to show some threshold $\sim r_n$

- draw grid of spacing εr_n
- random point “activates” closest grid point
- count relevant grid patterns

let’s implement this twice, 1 for torus, 2 for metric space

$nL_n^d \geq c \log n$: torus

- assume $\{L_n \leq r_n\}$ happens with $nr_n^d = c \log n$
- at each $\varepsilon_1 r_n$ lattice point, EITHER non-activated OR at least two points within $(1 + 2\sqrt{d}\varepsilon_1) r_n$

$$\mathbb{P}[. \cup . .] = 1 - \mathbb{P}[\{. \}^c | \{. . \}^c]\mathbb{P}[\{. . \}^c] \leq 1 - \eta \mathbb{P}[\{. . \}^c]$$

- keeps lattices distant $3r_n$ from each other, by independence

$$(1 - \ldots)^{c_1 r_n^{-d}} = O(\exp(-n^{1-\theta_d c})) \to 0$$

provided $c < 1/\theta_d$

$nL_n^d \geq c \log n$: metric space

- a portion of A is r-packed with packing number $\Omega(r^{-b})$
- $\mu(B(x,r)) \leq ar^d$ for every x in that portion

$$(1-\eta(nar^d)e^{-nar^d})^{c_1 r_n^{-b}} = O(\exp(-n^{b/d-ac})) \to 0$$

provided $c < (b/d)/a$

- the ultimate lower bound is the max of these $(b/d)/a$ for all kinds of portions

$nM_n^d \le c\log n$: torus

suppose $\{M_n > r_n\}$ with $nr_n^d = c\log n$, then $\sharp$ clusters ≥ 2, so one of the following happens

- $\exists$ isolated point
- $\exists$ non-isolated small clusters, $0 < diam \le Kr_n$
- $\exists$ two big clusters, $diam > Kr_n$

Implement granulation for each of the above cases. e.g. $\mathbb{P}[\exists iso] \le r_n^{-d}\exp(-c\theta_d \log n) = n^{1-c\theta_d} \to 0$ provided $c > 1/\theta_d$.

$nM_n^d \leq c \log n$: metric space

- a portion of A is covered by $O(r^{-b})$ balls with radius r and $\mu(B(x,r)) \geq ar^d$ for each x in that portion

- $\mathbb{P}[\exists iso \text{ in portion}] \leq O(r_n^{-b}) \exp(-nar_n^d) = n^{b/d-ac} \to 0$

  provided $c > (b/d)/a$.
- union bound: $c > \max(b/d)/a$ over all portions
- small and big clusters are more complicated, draw

provided that A has matching covering and packing exponent and one can estimate volume of balls ($G_r \setminus G$) optimally (up to epsilon), all three thresholds are asymptotically the same

Isolated points and AB coverage

Given $\mathcal{X}, \mathcal{Y}$ sampled from A

$$T = \inf\{r : \mathcal{X}(B(y,r)) \geq 1, \forall y \in \mathcal{Y}\}$$

- lower bound of R_n, the minimal matching threshold, and connectivity threshold of BRGG
- related to AB-percolation, Iyer-Yogeshwaren

$$n\mathbb{E}[|x \in A : \mathcal{P}_n(B(x,r)) = 0|] = \int_A \exp(-n\theta_d r^d) n dx = \mathbb{E}[$$

> $\mathcal{Y}(\text{uncovered}_{r_n}) \to Po(.)$ for some $r_n \Rightarrow T_n \xrightarrow{d} (2compt)Gumbel$

Papers

- **Largest nearest-neighbour link and connectivity threshold in a polytopal random sample** *Journal of Applied and Computational Topology*
- **Covering one point process with another** *submitted*

code:

https://github.com/frankiehiggs/connectivity-in-polytopes

https://github.com/frankiehiggs/CovXY

package:

https://github.com/xiaochuany/geography

Bonne anniversaire, Anne!