On geometric properties of anisotropic Gaussian random fields

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Anisotropic Analysis of Some Gaussian Models

Aline Bonami and Anne Estrade

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ABSTRACT. Although the classical Fractional Brownian Motion is often used to describe porosity, it is not adapted to anisotropic situations. In the present work, we study a class of Gaussian fields with stationary increments and "spectral density." They present asymptotic self-similarity properties and are good candidates to model a homogeneous anisotropic material, or its radiographic images. Unfortunately, the paths of all Gaussian fields with stationary increments have the same apparent regularity in all directions (except at most one). Hence we propose here a procedure to recover anisotropy from one realization: computing averages over all the hyperplanes which are orthogonal to a fixed direction, we get a process whose Hölder regularity depends explicitly on the asymptotic behavior of the spectral density in this direction.

Motivation and Introduction

Thirty years ago, Mandelbrot and Van-Ness [17] have initiated the description of 1-dimensional data through Fractional Brownian Motion (FBM). Since then, Fractal Analysis is often used for the description of roughness or porosity of some d-dimensional material.

1. Introduction

Let $X = \{X(t),\ t \in \mathbb{R}^N\}$ be a random field with values in $\mathbb{R}^d$ and let $T \subset \mathbb{R}^N$ be a compact interval.

We are interested in the following random sets:

- Range $X(T) = \{X(t) : t \in T\}$
- Graph $\mathrm{Gr}X\big(T\big) = \{(t, X(t)) : t \in T\}$
- Level set $X^{-1}(x) = \big\{t \in \mathbb{R}^N : X(t) = x\big\}$
- Excursion set $X^{-1}(F) = \big\{t \in \mathbb{R}^N : X(t) \in F\big\},\ \forall F \subseteq \mathbb{R}^d$.

  If $d = 1$ and $F = [u, \infty)$, then $X^{-1}(F) \cap T$ is the excursion set

  $$E_X(u) = \{t \in T : X(t) \geq u\}$$

  considered in Prof. Céline Duval's lectures.
- The set of self-intersections, ...

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The properties of these random sets depend on the smoothness or roughness of the sample function $X(t)$.

- If $X(t)$ is not smooth, one uses fractal geometry to study the random sets generated by X.
- If $X(t)$ is smooth, one uses integral geometry to characterize the topological structures of the random sets.

It is known that the expected Euler characteristic of the excursion set $E_X(u)$ is closely related to the exceedence probability

$$\mathbb{P}\left\{ \sup_{t \in T} X(t) > u \right\},$$

which is important in many applications.

Gaussian fields: some examples

As in Bonami and Estrade (2023), we consider a centered Gaussian random field $X = \{X(t),\ t \in \mathbb{R}^N\}$ with stationary increments and $X(0) = 0$. If $R(s,t) = \mathbb{E}\big[X(s)X(t)\big]$ is continuous, then $R(s,t)$ can be written as

$$R(s,t) = \int_{\mathbb{R}^N} (e^{i\langle s,\,\lambda\rangle} - 1)(e^{-i\langle t,\,\lambda\rangle} - 1)\Delta(d\lambda),$$

where $\Delta(d\lambda)$ is a Borel measure which satisfies

$$\int_{\mathbb{R}^N} \frac{|\lambda|^2}{1+|\lambda|^2}\, \Delta(d\lambda) < \infty. \tag{1}$$

The measure Δ is called the *spectral measure* of X.

It follows that X has the stochastic integral representation:

$$X(t) \stackrel{d}{=} \int_{\mathbb{R}^N} \left(e^{i\langle t,\lambda\rangle} - 1\right) \mathcal{M}(d\lambda),$$

where $\stackrel{d}{=}$ denotes equality of all finite-dimensional distributions, $\mathcal{M}(d\lambda)$ is a centered complex-valued Gaussian random measure with Δ as its control measure.

Example 1.1. If Δ has a density function

$$f_H(\lambda) = c(H,N)|\lambda|^{-(2H+N)},$$

where $H \in (0,1)$ and $c(H,N) > 0$ is a constant, then X is fractional Brownian motion with index H.

Example 1.2. A large class of Gaussian fields can be obtained by letting spectral density functions satisfy (1) and

$$f(\lambda) \asymp \frac{1}{\left(\sum_{j=1}^{N} |\lambda_j|^{\beta_j}\right)^{\gamma}}, \qquad \forall\, \lambda \in \mathbb{R}^N,\ |\lambda| \geq 1, \tag{2}$$

where $(\beta_1, \ldots, \beta_N) \in (0, \infty)^N$ and

$$\gamma > \sum_{j=1}^{N} \frac{1}{\beta_j}.$$

This last condition is necessary for $f \in L^2(\mathbb{R}^N)$.

More examples

The following anisotropic random fields do not have stationary increments.

- Fractional Brownian sheet $W^H = \left\{W^H(t), t \in \mathbb{R}^N\right\}$ is a mean 0 Gaussian field in $\mathbb{R}$ with covariance function

$$\mathbb{E}\left[W^H(s)W^H(t)\right] = \prod_{j=1}^{N} \frac{1}{2}\left(|s_j|^{2H_j} + |t_j|^{2H_j} - |s_j - t_j|^{2H_j}\right),$$

where $H = (H_1, \ldots, H_N) \in (0,1)^N$.
For all constants $c > 0$,

$$\left\{W^H(c^E t),\ t \in \mathbb{R}^N\right\} \stackrel{d}{=} \left\{c\, W^H(t),\ t \in \mathbb{R}^N\right\},$$

where $E = (a_{ij})$ is the $N \times N$ diagonal matrix with $a_{ii} = 1/(NH_i)$ for all $1 \le i \le N$ and $a_{ij} = 0$ if $i \neq j$.

- Solution to stochastic heat equation:

$$\frac{\partial u}{\partial t}(t,x) = \frac{\partial^2 u}{\partial x^2}(t,x) + \dot{W}(t,x),$$

where $t \geq 0$, $x \in \mathbb{R}$ and $\dot{W}(t,x)$ is a space-time white noise.

- Operator-scaling fields: Biermé, Meerschaert and Scheffler (2007).

2. Regularity properties

We consider a Gaussian field $X = \{X(t), t \in \mathbb{R}^N\}$ with stationary increments such that its spectral density satisfies (1) and

$$f(\lambda) \asymp \frac{1}{\left(\sum_{j=1}^N |\lambda_j|^{\beta_j}\right)^{\gamma}}, \qquad \forall\, \lambda \in \mathbb{R}^N,\ |\lambda| \geq 1, \tag{3}$$

where $(\beta_1, \ldots, \beta_N) \in (0, \infty)^N$ and

$$\gamma > \sum_{j=1}^N \frac{1}{\beta_j}$$

are constants.

A similar example for $N = 2$ was considered by Bonami and Estrade (2003). They established, among many other interesting results, the Hölder continuity of the sample functions of $X(t)$.

Theorem 2.1. [Xue and Xiao, 20011]

Let $X = \{X(t), t \in \mathbb{R}^N\}$ be a centered Gaussian field with stationary increments and spectral density satisfying (3).

(i) For any $j \in \{1, \ldots, N\}$, if

$$\beta_j\Big(\gamma - \sum_{i=1}^{N} \frac{1}{\beta_i}\Big) > 2, \tag{4}$$

then the partial derivative $\partial X(t)/\partial t_j$ is continuous almost surely. In particular, if (4) holds for all $1 \le j \le N$, then almost surely $X(t)$ is continuously differentiable.

(ii) If

$$\max_{1\le j\le N} \beta_j\Big(\gamma - \sum_{i=1}^{N} \frac{1}{\beta_i}\Big) \le 2, \tag{5}$$

then the sample function $X(t)$ is not differentiable in any direction.

Exact uniform modulus of continuity

Under condition (5) with strict inequality, we have

Theorem 2.2 [Meerschaert, Wang and X., 2013]

The exact modulus of continuity of $X(t)$ is given by

$$\limsup_{|h|\to 0} \frac{\sup_{t\in T, s\in[0,h]} |X(t+s) - X(t)|}{\rho(0,h)\sqrt{\log(1+\rho(0,h)^{-1})}} = \kappa. \qquad (6)$$

In the above, $\kappa \in (0,\infty)$ is a constant, ρ is the metric on $\mathbb{R}^N$ defined by

$$\rho(s,t) = \sum_{j=1}^{N} |s_j - t_j|^{H_j}, \qquad (7)$$

where for every $1 \le j \le N$,

$$H_j = \frac{\beta_j}{2}\Big(\gamma - \sum_{i=1}^{N} \frac{1}{\beta_i}\Big) \in (0,1). \qquad (8)$$

Meerschaert, Wang and X. (2013) proved (6) under the following general conditions:
Let $H = (H_1, \ldots, H_N) \in (0,1)^N$ be a constant vector. There exist positive and finite constants $c_1, \ldots, c_4$ such that

(A1) For all $s, t \in T$, $\mathbb{E}[X(t)^2] \geq c_1$ and

$$c_2 \rho(s,t)^2 \leq \mathbb{E}\left[(X(s) - X(t))^2\right] \leq c_3 \rho(s,t)^2.$$

(A2) For all $n \geq 1$ and $u, t^1, \ldots, t^n \in T$,

$$\mathrm{Var}\big(X(u) \,\big|\, X(t^1), \ldots, X(t^n)\big) \geq c_4 \sum_{j=1}^{N} \min_{1 \leq k \leq n} \left|u_j - t_j^k\right|^{2H_j},$$

where $\mathrm{Var}\big(X(u) \,\big|\, X(t^1), \ldots, X(t^n)\big)$ denotes the conditional variance of $X(u)$ given $X(t^1), \ldots, X(t^n)$.

Local oscillations

For comparison purpose, we also state the following results on the local oscillations. They are also useful for study some fractal properties of $X(t)$ [e.g., the set of "fast points"].

Theorem 2.3 [Lee and X. 2021]

Let $X = \{X(t), t \in \mathbb{R}^N\}$ be a centered Gaussian field that satisfies (A1) and (**B1**) below. Then for every $t^0 \in T$, there is a constant $\kappa_2 = \kappa_2(t^0) \in (0, \infty)$ such that

$$\limsup_{|h| \downarrow 0} \sup_{s \in [-h, h]} \frac{|X(t^0 + s) - X(t^0)|}{\varphi_1(s)} = \kappa_1, \qquad \text{a.s.,} \tag{9}$$

where

$$\varphi_1(s) = \rho(0, s) \left[\log\log \left(1 + \frac{1}{\prod_{j=1}^N |s_j|^{H_j}} \right) \right]^{\frac{1}{2}}, \qquad \forall\, s \in \mathbb{R}^N.$$

Theorem 2.4 [Lee and X. 2021]

Let $X = \{X(t), t \in \mathbb{R}^N\}$ be a centered Gaussian random field that satisfying conditions (A1), (A2) and (**B1**) below. Then for every $t^0 \in T$, there is a constant $\kappa_3 = \kappa_3(t^0) \in (0, \infty)$ such that

$$\liminf_{r \to 0} \frac{\max_{s:\rho(s,t^0)\le r} |X(t^0 + s) - X(t^0)|}{r(\log\log 1/r)^{-1/Q}} = \kappa_3, \qquad \text{a.s.} \tag{10}$$

Chung's LIL describes the smallest local oscillation of $X(t)$, which is useful for studying hitting probabilities and fractal properties of X.

3. Rough case: hitting probabilities and the Hausdorff dimension of $X^{-1}(F)$

Theorem 3.1 [Biermé, Lacaux and X. (2009)]

Let $X = \{X(t), t \in \mathbb{R}^N\}$ be a Gaussian field defined by

$$X(t) = \big(X_1(t), \ldots, X_d(t)\big), \qquad t \in \mathbb{R}^N, \tag{11}$$

where $X_1, \ldots, X_d$ are independent copies of a centered Gaussian field X_0 that satisfies Conditions (A1) and (A2) with $n = 1$. Then $\forall$ Borel set $F \subset \mathbb{R}^d$,

$$c_5\, \mathcal{C}^{d-Q}(F) \le \mathbb{P}\big\{X(T) \cap F \neq \emptyset\big\} \le c_6\, \mathcal{H}^{d-Q}(F), \tag{12}$$

where $Q = \sum_{j=1}^N \frac{1}{H_j}$, $\mathcal{C}^{d-Q}$ is $(d-Q)$-dimensional Riesz capacity and $\mathcal{H}^{d-Q}$ is $(d-Q)$-dimensional Hausdorff measure.

Remarks. Theorem 3.1 implies that

- If $\mathcal{C}^{d-Q}(F) > 0$, then $\mathbb{P}\{X(T) \cap F \neq \emptyset\} > 0$.
- If $\mathcal{H}^{d-Q}(F) = 0$, then $\mathbb{P}\{X(T) \cap F \neq \emptyset\} = 0$.

However, when $\mathcal{H}^{d-Q}(F) > 0$ (which holds when $d = Q$ and $F \neq \emptyset$), Theorem 1.1 is not informative.

It is an open problem if $\mathcal{H}^{d-Q}(F)$ in (12) can be replaced by $\mathcal{C}^{d-Q}(F)$, except in the following two cases:

- The Brownian sheet: Khoshnevisan and Shi (1999).
- The case F is a singleton: Dalang, Mueller and X. (2017): if $d = Q$, then for every $x \in \mathbb{R}^d$,

$$\mathbb{P}\{X(T) \cap \{x\} \neq \emptyset\} = \mathbb{P}\{\exists t \in T : X(t) = x\} = 0.$$

Hausdorff dimension of the excursion set

Theorem 3.2 [Biermé, Lacaux and X. (2009)]

Let $F \subseteq \mathbb{R}^d$ be a Borel set such that $\sum_{j=1}^{N} \frac{1}{H_j} > d - \dim F$. Then with positive probability,

$$\dim\Big(X^{-1}(F) \cap T\Big) = \min_{1 \le k \le N} \left\{ \sum_{j=1}^{k} \frac{H_k}{H_j} + N - k - H_k(d - \dim F) \right\}.$$

More properties such as uniform Hausdorff dimension result for $X^{-1}(F)$ or the exact Hausdorff measure of the $X^{-1}(x)$ can be obtained by studying the local times of X.

Theorem 3.1 was applied by Jaramillo and Nualart (2020) to studying the collision of eigenvalues of random matrices with Gaussian random field entries.

Song, X. and Yuan (2021) extended the work of Jaramillo and Nualart (2020) to multiple spectral collisions, and applied Theorem 3.2 to determine the Hausdorff dimension of the set of times of spectral collisions.

However, due to the lack of information in Theorem 3.1 in the critical case when

$$\mathcal{C}^{d-Q}(F) = 0 \quad \text{and} \quad \mathcal{H}^{d-Q}(F) > 0,$$

they were not able to solve the problem on the existence of spectral collisions completely.

This motivated Lee, Song, X. and Yuan (2023) to further study the hitting probability problem in the critical dimension case, and apply the result to solve the problem on collision of eigenvalues.

Let $X = \{X(t), t \in \mathbb{R}^N\}$ be a Gaussian field in $\mathbb{R}^d$ defined (11). We use the setting in Dalang, Mueller and X. (2017).

The conditions **(B1)-(B3)** are formulated in a general way so that they cover many Gaussian random fields and solutions to SPDEs.

They are satisfied by a fractional Brownian field of index $H \in (0, 1)$:

$$B^H(t) = \int_{\mathbb{R}^N} (e^{i\langle t,x\rangle} - 1)\frac{\widetilde{W}(dx)}{|x|^{H+\frac{N}{2}}},$$

where $\widetilde{W}$ is a complex-valued Gaussian random measure with Lebesgue control measure.

(B1) There is a Gaussian random field $\{W(A,t) : A \in \mathcal{B}(\mathbb{R}_+), t \in \mathbb{R}^N\}$ satisfying the following two conditions:

(b1) For all $t \in \mathbb{R}^N$, $A \mapsto W(A,t)$ is an $\mathbb{R}^d$-valued Gaussian noise with a control measure ν_t such that $W(\mathbb{R}_+, t) = X(t)$ and when $A \cap B = \emptyset$, $W(A, \cdot)$ and $W(B, \cdot)$ are independent.

(b2) $\exists\ a_0 \geq 0$, $c_7 > 0$, $\gamma_j > 0$, $j = 1, \ldots, N$, such that for all $a_0 \leq a < b \leq +\infty$ and all $s := (s_1, \ldots, s_N), t := (t_1, \ldots, t_N) \in T$ (a compact interval),

$$\big\| W([a,b), s) - X(s) - W([a,b), t) + X(t) \big\|_{L^2} \leq c_7 \Big[\sum_{j=1}^N a^{\gamma_j} |s_j - t_j| + b^{-1} \Big],$$

$$\big\| W([0,a_0), s) - W([0,a_0), t) \big\|_{L^2} \leq c_7 \sum_{j=1}^N |s_j - t_j|,$$

where $\|Y\|_{L^2} := \big(\mathbb{E}[Y_1^2 + \cdots + Y_d^2]\big)^{1/2}$.

Denote

$$H_j = (1 + \gamma_j)^{-1}, \quad 1 \le j \le N. \tag{13}$$

These parameters are useful for characterizing various properties of the random field X.

The following lemma bounds the canonical metric induced by $\|X(s) - X(t)\|_{L^2}$ by using the metric ρ.

Lemma 3.3

Under Assumption **(B1)**, for all $s, t \in T$ with $\rho(s,t) \le \min\{a_0^{-1}, 1\}$, we have

$$\big\|X(s) - X(t)\big\|_{L^2} \le 4c_4 \rho(s,t).$$

Condition (**B1**) indicates that $X(t)$ can be approximated by $W([a,b],t)$. The following lemma quantifies the approximation error.

Lemma 3.4 [Dalang, Mueller, X. (2017)]

Assume that (**B1**) holds. For $b > a > 1$ and $r > 0$, set

$$A = \sum_{j=1}^{N} a^{H_j^{-1}-1}\, r^{H_j^{-1}} + b^{-1}.$$

There are constants A_0, K and c such that for $A \le A_0 r$ and

$$u \ge KA \log^{1/2}\left(\frac{r}{A}\right),$$

we have for all $t^0 \in T$,

$$\mathbb{P}\left\{ \sup_{t \in S(t^0, r)} \left| X(t) - X(t^0) - (W([a,b],t) - W([a,b],t^0)) \right| \ge u \right\}$$

$$\le \exp\left(-\frac{u^2}{cA^2}\right),$$

where $S(t^0, r) = \{t \in T : \rho(t,\, t^0) \le r\}$.

We further impose the following two assumptions on X.

(B2) $\exists$ a constant $c_8 > 0$, such that $\|X_i(t)\|_{L^2} \geq c_8$ for all $t \in T^{(\epsilon_0)}$ (the ϵ_0-neighborhood of T) and all $1 \leq i \leq N$.

(B3) $\exists$ a constant $\rho_0 > 0$ with the following property. For $t \in T$, there exist $t' = t'(t) \in T^{(\epsilon_0)}$, $\delta_j = \delta_j(t) \in (H_j, 1]$ for $1 \leq i \leq N$ and $C = C(t) > 0$, such that

$$\left| \mathbb{E}\left[X_i(t')\big(X_i(s) - X_i(\bar{s})\big)\right] \right| \leq C \sum_{j=1}^{N} |s_j - \bar{s}_j|^{\delta_j},$$

for all $1 \leq i \leq N$ and all $s, \bar{s} \in T^{(\epsilon_0)}$ with

$$\max\{\rho(t,s),\ \rho(t,\bar{s})\} \leq 2\rho_0.$$

The following is the main result.

Theorem 3.5 [Lee, Song, X. and Yuan (2023)]

Assume **(B1)** - **(B3)** hold and suppose $d \geq Q$. Let $F \subset \mathbb{R}^d$ be a bounded set that satisfies the following condition: $\exists$ constants $\theta \in [0, d-Q]$, $C_F \in (0, \infty)$, and $\kappa \in [0, (d-\theta)/Q)$ such that

$$\lambda_d(F^{(r)}) \leq C_F r^{d-\theta} \big(\log\log(1/r) \big)^{\kappa} \tag{14}$$

for all $r > 0$ small, where λ_d is the Lebesgue measure on $\mathbb{R}^d$ and $F^{(r)} = \big\{x \in \mathbb{R}^d : \inf_{y \in F} |x-y| \leq r\big\}$ is the (closed) r-neighborhood of F. Then

$$X^{-1}(F) \cap T = \emptyset, \ \text{a.s.}$$

Remark. Eq. (14) implies that $\overline{\dim}_{\rm M} F \leq \theta$, where $\overline{\dim}_{\rm M}$ denotes the upper Minkowski dimension, and allows F to have positive θ-dimensional Hausdorff measure.

The following corollary shows two cases that could not be handled by Theorem 3.1.

Corollary 3.6

Assume **(B1)-(B3)** hold and $F \subset \mathbb{R}^d$ is a bounded set.

(i) If $d > Q$, $\dim F = d - Q$ and (14) holds with $\theta = d - Q$ and a constant $\kappa < 1$, then $X^{-1}(F) \cap T = \emptyset$ a.s.

(ii) If $d = Q$ and F satisfies (14) with $\theta = 0$ and a constant $\kappa < 1$, then $X^{-1}(F) \cap T = \emptyset$ a.s.

In particular, (ii) extends the result for singleton $F = \{x\}$ in Dalang, Mueller and X. (2017) to uncountable infinite sets.

The key ingredient for proving Theorem 3.5 is the following proposition proved in Dalang, Mueller and X. (2017), which is analogous to Proposition 4.1 of Talagrand (1995).

Proposition 3.7

Assume **(B1)** hold. Then there exist constants $K \in (0, \infty)$ and $\delta_0 \in (0, 1]$ such that for any $r_0 \in (0, \delta_0)$ and $t \in T$,

$$\mathbb{P}\Big\{\exists r \in [r_0^2, r_0], \sup_{s \in T: \rho(s,t) < r} |X(s) - X(t)| \le K\, r \Big(\log\log 1/r\Big)^{-1/Q}\Big\}$$
$$\ge 1 - \exp\Big(-\sqrt{\log 1/r_0}\Big).$$

Based on Proposition 3.7 and other properties of the Gaussian random field X, we can construct an economic random covering for $X^{-1}(F) \cap T$ and prove Theorem 3.5.

4. The mean Euler characteristic of $E_X(u)$

When the sample function $X(\cdot) \in C^2(\mathbb{R}^N)$ and is a Morse function a.s., Cheng and Xiao (2016) computed the expected Euler characteristic of $E_X(u)$:

$$\mathbb{E}\{\varphi(E_X(u))\}$$

$$= \sum_{\{t\} \in \partial_0 T} \mathbb{P}(X(t) \geq u, \nabla X(t) \in E(\{t\})) + \sum_{k=1}^{N} \sum_{J \in \partial_k T} \frac{1}{(2\pi)^{k/2} |\Lambda_J|^{1/2}}$$

$$\times \int_J dt \int_u^\infty dx \int \cdots \int_{E(J)} dy_{J_1} \cdots dy_{J_{N-k}} \frac{|\Lambda_J - \Lambda_J(t)|}{\gamma_t^k}$$

$$\times H_k\Big(\frac{x}{\gamma_t} + \gamma_t C_{J_1}(t) y_{J_1} + \cdots + \gamma_t C_{J_{N-k}}(t) y_{J_{N-k}}\Big)$$

$$\times p_{X(t), X_{J_1}(t), \ldots, X_{J_{N-k}}(t)}(x, y_{J_1}, \ldots, y_{J_{N-k}} | \nabla X_{|J}(t) = 0),$$

and prove that it approximates the excursion probability:

$$\mathbb{P}\Big\{\sup_{t \in T} X(t) \geq u\Big\} = \mathbb{E}\{\varphi(E_X(u))\}\big(1 + o\big(e^{-\alpha u^2}\big)\big), \quad \text{as } u \to \infty.$$

Thank you for your attention!

Happy birthday, Anne!