Fractional operators and fractionally integrated random fields on $\mathbb{Z}^\nu$

Vytautė Pilipauskaitė (Aalborg University, Denmark)

joint work with Donatas Surgailis (Vilnius University, Lithuania)

2024-05-30

Outline

1. Introduction: $\mathbb{Z}$
2. Fractional integration on $\mathbb{Z}^\nu$
3. Fractionally integrated random fields X on $\mathbb{Z}^\nu$
4. Scaling limits of X

1. Introduction

Denote all functions $g : \mathbb{Z} \to \mathbb{R}$ by $S(\mathbb{Z})$ and consider operators from $S(\mathbb{Z})$ to $S(\mathbb{Z})$:

- backshift

$$Tg(t) = g(t-1)$$

1. Introduction

Denote all functions $g : \mathbb{Z} \to \mathbb{R}$ by $S(\mathbb{Z})$ and consider operators from $S(\mathbb{Z})$ to $S(\mathbb{Z})$:

- backshift

$$Tg(t) = g(t-1)$$

- its jth power

$$T^j g(t) = g(t-j)$$

  with $T^0 = I$ identity, $T^1 = T$

1. Introduction

Denote all functions $g : \mathbb{Z} \to \mathbb{R}$ by $S(\mathbb{Z})$ and consider operators from $S(\mathbb{Z})$ to $S(\mathbb{Z})$:

- backshift

$$Tg(t) = g(t-1)$$

- its jth power

$$T^j g(t) = g(t-j)$$

  with $T^0 = I$ identity, $T^1 = T$

- its polynomial, e.g., discrete derivative

$$(I - T)g(t) = g(t) - g(t-1)$$

For $d \in (-1, 1)$, $d \neq 0$, the fractional operator is defined by

$$\boxed{(I - T)^d g(t) = \sum_{j=0}^{\infty} \psi_j(d) T^j g(t)} = \sum_{j=0}^{\infty} \psi_j(d) g(t - j)$$

For $d \in (-1, 1)$, $d \neq 0$, the fractional operator is defined by

$$\boxed{(I - T)^d g(t) = \sum_{j=0}^{\infty} \psi_j(d) T^j g(t)} = \sum_{j=0}^{\infty} \psi_j(d) g(t - j)$$

through binomial expansion

$$(1 - z)^d = \sum_{j=0}^{\infty} \psi_j(d) z^j, \quad z \in \mathbb{C},\ |z| < 1,$$

where

$$\psi_j(d) = \frac{\Gamma(j - d)}{\Gamma(j + 1)\Gamma(-d)} = \prod_{0 < i \leq j} \frac{i - 1 - d}{i}, \quad j = 0, 1, 2, \ldots$$

Let $d \in (-1/2, 1/2)$, $d \neq 0$ and $\{\varepsilon(t),\, t \in \mathbb{Z}\}$ be a white noise, i.e. a sequence of random variables such that

$$\mathbb{E}[\varepsilon(t)] = 0, \quad \mathbb{E}[\varepsilon(t)\varepsilon(s)] = \mathbf{1}(t = s), \quad t, s \in \mathbb{Z}.$$

Then

$$X(t) = (I - T)^{-d}\varepsilon(t), \quad t \in \mathbb{Z},$$

Let $d \in (-1/2, 1/2)$, $d \neq 0$ and $\{\varepsilon(t),\, t \in \mathbb{Z}\}$ be a white noise, i.e. a sequence of random variables such that

$$\mathbb{E}[\varepsilon(t)] = 0, \quad \mathbb{E}[\varepsilon(t)\varepsilon(s)] = \mathbf{1}(t = s), \quad t, s \in \mathbb{Z}.$$

Then

$$X(t) = (I - T)^{-d} \varepsilon(t), \quad t \in \mathbb{Z},$$

is an ARFIMA$(0, d, 0)$ process, i.e. a stationary solution of the equation

$$(I - T)^{d} X(t) = \varepsilon(t), \quad t \in \mathbb{Z}.$$

Let $d \in (-1/2, 1/2)$, $d \neq 0$ and $\{\varepsilon(t),\, t \in \mathbb{Z}\}$ be a white noise, i.e. a sequence of random variables such that

$$\mathbb{E}[\varepsilon(t)] = 0, \quad \mathbb{E}[\varepsilon(t)\varepsilon(s)] = \mathbf{1}(t = s), \quad t, s \in \mathbb{Z}.$$

Then

$$X(t) = (I - T)^{-d}\varepsilon(t), \quad t \in \mathbb{Z},$$

is an ARFIMA$(0, d, 0)$ process, i.e. a stationary solution of the equation

$$(I - T)^{d} X(t) = \varepsilon(t), \quad t \in \mathbb{Z}.$$

- Granger, Joyeux 1980, Hosking 1981
- Convergence of series of random variables in mean square
- Long-range/negative dependence

Problem

Extend fractional operators

$$(I - T)^d = \sum_{j=0}^{\infty} \psi_j(d) T^j, \quad d \in (-1, 1),\ d \neq 0,$$

to more general $T : S(\mathbb{Z}^\nu) \to S(\mathbb{Z}^\nu)$ for $\nu \geq 1$.

2. Fractional integration on $\mathbb{Z}^\nu$

Let $\{S_j,\, j = 0, 1, \dots\}$ be a random walk on $\mathbb{Z}^\nu$ with $S_0 = \mathbf{0}$ and

$$p(\boldsymbol{s}) = \mathbb{P}(S_1 = \boldsymbol{s}), \quad \boldsymbol{s} \in \mathbb{Z}^\nu.$$

2. Fractional integration on $\mathbb{Z}^\nu$

Let $\{S_j,\, j = 0, 1, \dots\}$ be a random walk on $\mathbb{Z}^\nu$ with $S_0 = \mathbf{0}$ and

$$p(\boldsymbol{s}) = \mathbb{P}(S_1 = \boldsymbol{s}), \quad \boldsymbol{s} \in \mathbb{Z}^\nu.$$

We introduce

$$\boxed{Tg(\boldsymbol{t}) = \sum_{\boldsymbol{s} \in \mathbb{Z}^\nu} p(\boldsymbol{s}) g(\boldsymbol{s} + \boldsymbol{t})} = \mathbb{E}[g(S_1 + \boldsymbol{t})], \quad \boldsymbol{t} \in \mathbb{Z}^\nu.$$

2. Fractional integration on $\mathbb{Z}^\nu$

Let $\{S_j,\, j = 0, 1, \dots\}$ be a random walk on $\mathbb{Z}^\nu$ with $S_0 = \mathbf{0}$ and

$$p(\boldsymbol{s}) = \mathbb{P}(S_1 = \boldsymbol{s}), \quad \boldsymbol{s} \in \mathbb{Z}^\nu.$$

We introduce

$$\boxed{Tg(\boldsymbol{t}) = \sum_{\boldsymbol{s} \in \mathbb{Z}^\nu} p(\boldsymbol{s}) g(\boldsymbol{s} + \boldsymbol{t})} = \mathbb{E}[g(S_1 + \boldsymbol{t})], \quad \boldsymbol{t} \in \mathbb{Z}^\nu.$$

Example 1

$(I - T)^d$ in dimension $\nu = 1$ in ARFIMA$(0, d, 0)$ with

$$Tg(t) = g(t-1), \quad t \in \mathbb{Z},$$

corresponds to

$$p(-1) = 1.$$

Let $\{S_j,\, j=0,1,\dots\}$ be a random walk on $\mathbb{Z}^\nu$ with $S_0=\mathbf{0}$ and

$$p(\boldsymbol{s}) = \mathbb{P}(S_1=\boldsymbol{s}), \quad \boldsymbol{s}\in\mathbb{Z}^\nu.$$

We introduce

$$\boxed{Tg(\boldsymbol{t}) = \sum_{\boldsymbol{s}\in\mathbb{Z}^\nu} p(\boldsymbol{s})g(\boldsymbol{s}+\boldsymbol{t})} = \mathbb{E}[g(S_1+\boldsymbol{t})], \quad \boldsymbol{t}\in\mathbb{Z}^\nu.$$

Example 2

Fractional Laplacian $(I-T)^d$ in dimension $\nu\ge 1$ with

$$Tg(\boldsymbol{t}) = \frac{1}{2\nu}\sum_{i=1}^{\nu}(g(\boldsymbol{t}-\boldsymbol{e}_i)+g(\boldsymbol{t}+\boldsymbol{e}_i)), \quad \boldsymbol{t}\in\mathbb{Z}^\nu,$$

corresponds to

$$p(\pm\boldsymbol{e}_i) = \frac{1}{2\nu}, \quad i=1,\dots,\nu,$$

where $\boldsymbol{e}_i\in\mathbb{Z}^\nu$ has 1 in the ith coordinate and 0's elsewhere.

Let $\{S_j,\, j = 0, 1, \ldots\}$ be a random walk on $\mathbb{Z}^\nu$ with $S_0 = \mathbf{0}$ and

$$p(\boldsymbol{s}) = \mathbb{P}(S_1 = \boldsymbol{s}), \quad \boldsymbol{s} \in \mathbb{Z}^\nu.$$

We introduce

$$\boxed{Tg(\boldsymbol{t}) = \sum_{\boldsymbol{s} \in \mathbb{Z}^\nu} p(\boldsymbol{s}) g(\boldsymbol{s} + \boldsymbol{t})} = \mathbb{E}[g(S_1 + \boldsymbol{t})], \quad \boldsymbol{t} \in \mathbb{Z}^\nu.$$

Example 3

For $\theta \in (0,1)$, fractional heat operator $(I - T)^d$ in dimension $\nu \geq 2$ with

$$Tg(\boldsymbol{t}) = (1-\theta) g(\boldsymbol{t} - \boldsymbol{e}_1) + \frac{\theta}{2(\nu-1)} \sum_{i=2}^{\nu} (g(\boldsymbol{t} - \boldsymbol{e}_1 - \boldsymbol{e}_i) + g(\boldsymbol{t} - \boldsymbol{e}_1 + \boldsymbol{e}_i)), \quad \boldsymbol{t} \in \mathbb{Z}^\nu,$$

corresponds to

$$p(-\boldsymbol{e}_1) = 1 - \theta, \quad p(-\boldsymbol{e}_1 \pm \boldsymbol{e}_i) = \frac{\theta}{2(\nu-1)}, \quad i = 2, \ldots, \nu - 1.$$

Since $p_j(\boldsymbol{s}) = \mathbb{P}(S_j = \boldsymbol{s})$ satisfies

$$p_j(\boldsymbol{s}) = \sum_{\boldsymbol{r}\in\mathbb{Z}^\nu} p(\boldsymbol{r})p_{j-1}(\boldsymbol{s}-\boldsymbol{r}), \quad \boldsymbol{s} \in \mathbb{Z}^\nu,$$

we have

$$T^j g(\boldsymbol{t}) = \sum_{\boldsymbol{s}\in\mathbb{Z}^\nu} p_j(\boldsymbol{s})g(\boldsymbol{s}+\boldsymbol{t}) = \mathbb{E}[g(S_j+\boldsymbol{t})], \quad \boldsymbol{t} \in \mathbb{Z}^\nu,$$

in

$$(I-T)^d g(\boldsymbol{t}) = \sum_{j=0}^{\infty} \psi_j(d) T^j g(\boldsymbol{t})$$

Since $p_j(\boldsymbol{s}) = \mathbb{P}(S_j = \boldsymbol{s})$ satisfies

$$p_j(\boldsymbol{s}) = \sum_{\boldsymbol{r} \in \mathbb{Z}^\nu} p(\boldsymbol{r}) p_{j-1}(\boldsymbol{s} - \boldsymbol{r}), \quad \boldsymbol{s} \in \mathbb{Z}^\nu,$$

we have

$$T^j g(\boldsymbol{t}) = \sum_{\boldsymbol{s} \in \mathbb{Z}^\nu} p_j(\boldsymbol{s}) g(\boldsymbol{s} + \boldsymbol{t}) = \mathbb{E}[g(S_j + \boldsymbol{t})], \quad \boldsymbol{t} \in \mathbb{Z}^\nu,$$

in

$$(I - T)^d g(\boldsymbol{t}) = \sum_{j=0}^{\infty} \psi_j(d) T^j g(\boldsymbol{t}) = \sum_{\boldsymbol{s} \in \mathbb{Z}^\nu} \tau(\boldsymbol{s}; d) g(\boldsymbol{s} + \boldsymbol{t}), \quad \boldsymbol{t} \in \mathbb{Z}^\nu.$$

where we then study fractional coefficients

$$\tau(\boldsymbol{s}; d) = \sum_{j=0}^{\infty} \psi_j(d) p_j(\boldsymbol{s}), \quad \boldsymbol{s} \in \mathbb{Z}^\nu.$$

Properties of binomial coefficients

Let $d \in (-1, 1)$, $d \neq 0$.

- Then by application of Stirling's formula,

$$\psi_j(d) \sim \Gamma(-d)^{-1} j^{-d-1}, \quad j \to \infty.$$

- If $d > 0$, then

$$\psi_j(d) < 0 \quad \text{for all } j = 1, 2, \ldots, \qquad \sum_{j=0}^{\infty} \psi_j(d) = 0.$$

- If $d < 0$, then

$$\psi_j(d) > 0 \quad \text{for all } j = 1, 2, \ldots, \qquad \sum_{j=0}^{\infty} \psi_j(d) = \infty.$$

Lemma (P., Surgailis 2024+)

- Let $0 < d < 1$. Then $0 < \tau(\mathbf{0}; d) < \infty$ and $-\infty < \tau(\boldsymbol{s}; d) \le 0$, $\boldsymbol{s} \ne \mathbf{0}$, and
$$\sum_{\boldsymbol{u} \in \mathbb{Z}^\nu} \tau(\boldsymbol{s}; d) = 0.$$
- Let $-1 < d < 0$. Then $0 \le \tau(\boldsymbol{s}; d) \le \infty$, $\boldsymbol{s} \in \mathbb{Z}^\nu$, and
$$\sum_{\boldsymbol{s} \in \mathbb{Z}^\nu} \tau(\boldsymbol{s}; d) = \infty.$$
- Let $0 < |d| < 1$ and $\tau(\mathbf{0}; -|d|) < \infty$. Then $(I - T)^d (I - T)^{-d} = I$.

Proposition (P., Surgailis 2024+)

Let $\{S_j,\ j = 0, 1, \dots\}$ be irreducible, aperiodic, $\mathbb{E}[S_1] = \mathbf{0}$, $\Gamma := \mathbb{E}[S_1 S_1^\top]$ and $\mathbb{E}\exp(c|S_1|) < \infty$ for some $c > 0$. Then for $-((\nu/2) \wedge 1) < d < 1$, $d \neq 0$, the fractional coefficients are well-defined and satisfy

$$\tau(\boldsymbol{s}; d) = (C + o(1))\langle \boldsymbol{s}, \Gamma^{-1}\boldsymbol{s}\rangle^{-(\nu/2)-d}, \qquad |\boldsymbol{s}| \to \infty,$$

where C is an explicit finite constant.

Proposition (P., Surgailis 2024+)

Let $\{S_j,\, j = 0, 1, \dots\}$ be irreducible, aperiodic, $\mathbb{E}[S_1] = \mathbf{0}$, $\Gamma := \mathbb{E}[S_1 S_1^\top]$ and $\mathbb{E}\exp(c|S_1|) < \infty$ for some $c > 0$. Then for $-((\nu/2) \wedge 1) < d < 1$, $d \neq 0$, the fractional coefficients are well-defined and satisfy

$$\tau(\boldsymbol{s}; d) = (C + o(1))\langle \boldsymbol{s}, \Gamma^{-1}\boldsymbol{s}\rangle^{-(\nu/2)-d}, \qquad |\boldsymbol{s}| \to \infty,$$

where C is an explicit finite constant.

- Proof uses the local CLT for a random walk from Lawler, Limic 2012

Proposition (P., Surgailis 2024+)

Let $\{S_j,\, j = 0, 1, \dots\}$ be irreducible, aperiodic, $\mathbb{E}[S_1] = \mathbf{0}$, $\Gamma := \mathbb{E}[S_1 S_1^\top]$ and $\mathbb{E}\exp(c|S_1|) < \infty$ for some $c > 0$. Then for $-((\nu/2) \wedge 1) < d < 1$, $d \neq 0$, the fractional coefficients are well-defined and satisfy

$$\tau(\boldsymbol{s}; d) = (C + o(1))\langle \boldsymbol{s}, \Gamma^{-1}\boldsymbol{s}\rangle^{-(\nu/2)-d}, \qquad |\boldsymbol{s}| \to \infty,$$

where C is an explicit finite constant.

- Proof uses the local CLT for a random walk from Lawler, Limic 2012
- Examples 2, 3 by similar arguments (see Koul, Mimoto, Surgailis 2016, P., Surgailis 2017, Surgailis 2020 for $\nu = 2$)

Denote the characteristic function

$$\widehat{p}(\boldsymbol{x}) = \mathbb{E} \exp\{\mathrm{i}\langle \boldsymbol{x}, S_1 \rangle\}, \quad \boldsymbol{x} \in \mathbb{R}^\nu.$$

Theorem (P., Surgailis 2024+)

For $d \in (-1, 1)$,

$$\int_{[-\pi,\pi]^\nu} |1 - \widehat{p}(\boldsymbol{x})|^{-2|d|} \mathrm{d}\boldsymbol{x} < \infty \iff \sum_{\boldsymbol{s} \in \mathbb{Z}^\nu} \tau(\boldsymbol{s}; -|d|)^2 < \infty. \tag{1}$$

Either of them implies that the Fourier transform

$$\widehat{\tau}(\,\cdot\,; -|d|) = (1 - \widehat{p}(\,\cdot\,))^{-|d|} \in L^2([-\pi, \pi]^\nu).$$

Moreover, for $d \in (0, 1)$, the above conditions hold with d in place of $-|d|$.

Proof uses approximation with

$$\tau_r(\boldsymbol{s}; d) = \sum_{j=0}^{\infty} r^j \psi_j(d) p_j(\boldsymbol{s}), \quad \boldsymbol{s} \in \mathbb{Z}^\nu, \quad r \in (0, 1).$$

3. Fractionally integrated random fields X on $\mathbb{Z}^\nu$

Corollary (P., Surgailis 2024+)

Let $d \in (-1, 1)$, $d \neq 0$ and (1) hold. Let $\{\varepsilon(\boldsymbol{t}), \boldsymbol{t} \in \mathbb{Z}^\nu\}$ be a white noise, i.e.

$$\mathbb{E}[\varepsilon(\boldsymbol{t})] = 0, \quad \mathbb{E}[\varepsilon(\boldsymbol{t})\varepsilon(\boldsymbol{s})] = \mathbf{1}(\boldsymbol{t} = \boldsymbol{s}), \quad \boldsymbol{t}, \boldsymbol{s} \in \mathbb{Z}^\nu.$$

Then

$$X(\boldsymbol{t}) = (I - T)^{-d} \varepsilon(\boldsymbol{t}), \quad \boldsymbol{t} \in \mathbb{Z}^\nu, \tag{2}$$

is a stationary solution of the equation

$$(I - T)^d X(\boldsymbol{t}) = \varepsilon(\boldsymbol{t}), \quad \boldsymbol{t} \in \mathbb{Z}^\nu,$$

where the both series converges in mean square.

Corollary (continued)

Denote

$$r(\boldsymbol{t}) = \mathrm{Cov}(X(\mathbf{0}), X(\boldsymbol{t})) = \sum_{\boldsymbol{s}\in\mathbb{Z}^\nu} \tau(\boldsymbol{s}; -d)\tau(\boldsymbol{s}+\boldsymbol{t}; -d), \quad \boldsymbol{t} \in \mathbb{Z}^\nu.$$

- If $d \in (0,1)$, then X has $r(\boldsymbol{t}) \geq 0$ and long-range dependence:

$$\sum_{\boldsymbol{t}\in\mathbb{Z}^\nu} |r(\boldsymbol{t})| = \infty.$$

- If $d \in (-1,0)$, then X has negative dependence:

$$\sum_{\boldsymbol{t}\in\mathbb{Z}^\nu} r(\boldsymbol{t}) = 0.$$

$\Longrightarrow$ Existence of the solution X in (2) for Examples 2, 3

For $d \in (-1/2, 1/2)$, $d \neq 0$, ARFIMA(0,d,0) process:

$$X(t) = \sum_{j=1}^{\infty} \psi_j(d) X(t-j) + \varepsilon(t) = \sum_{j=0}^{\infty} \psi_j(-d) \varepsilon(t-j) \quad t \in \mathbb{Z}.$$

- The first series is the best linear predictor (or conditional expectation in the Gaussian case) of $X(t)$ given the 'past' $X(s)$ since $\mathrm{Cov}(X(s), \varepsilon(t)) = 0$, $s < t$
- CAR random fields: Besag 1974, 1995 and references therein
- CAR random fields with long-range dependence: Ferretti, Ippoliti, Valentini, Bhansali 2023

Corollary (P., Surgailis 2024+)

Let $d \in (-1,1)$, $d \neq 0$ and (1) holds. Then X given by (2) has a representation:

$$X(\boldsymbol{t}) = \sum_{\boldsymbol{s} \neq \boldsymbol{0}} b(\boldsymbol{s}) X(\boldsymbol{t}-\boldsymbol{s}) + \zeta(\boldsymbol{t}), \quad \boldsymbol{t} \in \mathbb{Z}^\nu,$$

where the series converges in mean square,

- $b(\boldsymbol{t}) = -\tau(-\boldsymbol{t}; d)/\tau(\boldsymbol{0}; d)$ and $\zeta(\boldsymbol{t}) = \varepsilon(\boldsymbol{t})/\tau(\boldsymbol{0}; d)$ such that

$$\mathrm{Cov}(\zeta(\boldsymbol{t}), \zeta(\boldsymbol{s})) = 0, \quad \boldsymbol{t} \neq \boldsymbol{s}.$$

- $b(\boldsymbol{t}) = -\gamma^*(\boldsymbol{t})/\gamma^*(\boldsymbol{0})$ with $\gamma^*(\boldsymbol{t}) = (2\pi)^{-\nu} \int_{\Pi^\nu} \exp(-\mathrm{i}\langle \boldsymbol{t}, \boldsymbol{x}\rangle)|1-\widehat{p}(\boldsymbol{x})|^{2d} \mathrm{d}\boldsymbol{x}$ and $\zeta(\boldsymbol{t}) = \gamma^*(\boldsymbol{0})^{-1} \int_{\Pi^\nu} \exp(\mathrm{i}\langle \boldsymbol{t}, \boldsymbol{x}\rangle)(1-\widehat{p}(-\boldsymbol{x}))^d Z(\mathrm{d}\boldsymbol{x})$, where $Z(\mathrm{d}\boldsymbol{x})$ is a complex-valued random measure on $\Pi^\nu = [-\pi,\pi]^\nu$ with zero mean, $\mathbb{E}|Z(\mathrm{d}\boldsymbol{x})|^2 = \mathrm{d}\boldsymbol{x}/(2\pi)^\nu$ satisfying $\varepsilon(\boldsymbol{t}) = \int_{\Pi^\nu} \exp(\mathrm{i}\langle \boldsymbol{t}, \boldsymbol{x}\rangle) Z(\mathrm{d}\boldsymbol{x})$, $\boldsymbol{t} \in \mathbb{Z}^\nu$, such that

$$\mathrm{Cov}(\zeta(\boldsymbol{t}), X(\boldsymbol{s})) = 0, \quad \boldsymbol{t} \neq \boldsymbol{s},$$

  if $d \in (0,1)$.

4. Scaling limits of X

For $\phi : \mathbb{R}^\nu \to \mathbb{R}$, consider the distribution of

$$\int_{\mathbb{R}^\nu} X([\boldsymbol{t}])\phi(\boldsymbol{t}/\lambda)\mathrm{d}\boldsymbol{t} \quad \text{as } \lambda \to \infty,$$

where

$$X(\boldsymbol{t}) = \sum_{\boldsymbol{s}\in\mathbb{Z}^\nu} a(\boldsymbol{t}-\boldsymbol{s})\varepsilon(\boldsymbol{s}), \quad \boldsymbol{t} \in \mathbb{Z}^\nu, \tag{3}$$

with IID random variables $\varepsilon(\boldsymbol{t})$, $\boldsymbol{t} \in \mathbb{Z}^\nu$, $\mathbb{E}[\varepsilon(\mathbf{0})] = 0$, $\mathbb{E}[|\varepsilon(\mathbf{0})|^2] = 1$ and $a \in S(\mathbb{Z}^\nu)$.

For example,

- $\phi(\boldsymbol{t}) = \mathbf{1}(\boldsymbol{t} \in [0,1]^\nu)$
- $a(\boldsymbol{t}) = \tau(-\boldsymbol{t}; -d)$ in our Example 2

Assumption A(d)

Let $a \in S(\mathbb{Z}^\nu)$ satisfy:

- if $d \neq 0$, then
$$a(\boldsymbol{t}) = |\boldsymbol{t}|^{2d-\nu}(\ell(\boldsymbol{t}/|\boldsymbol{t}|) + o(1)), \quad |\boldsymbol{t}| \to \infty,$$
where $\ell \neq 0$ is a continuous real-valued function on $\{\boldsymbol{t} \in \mathbb{R}^\nu : |\boldsymbol{t}| = 1\}$, moreover, if $d < 0$, then
$$\sum_{\boldsymbol{t} \in \mathbb{Z}^\nu} a(\boldsymbol{t}) = 0.$$
- if $d = 0$, then
$$\sum_{\boldsymbol{t} \in \mathbb{Z}^\nu} |a(\boldsymbol{t})| < \infty, \quad \sum_{\boldsymbol{t} \in \mathbb{Z}^\nu} a(\boldsymbol{t}) \neq 0.$$

Note that X has long-range dependence if $d > 0$, negative dependence if $d < 0$ and short-range dependence if $d = 0$.

For $\phi : \mathbb{R}^\nu \to \mathbb{R}$, we define

$$W(\phi) = \begin{cases} \int_{\mathbb{R}^\nu} \left(\int_{\mathbb{R}^\nu} a_\infty(\boldsymbol{t} - \boldsymbol{s}) \phi(\boldsymbol{t}) \mathrm{d}\boldsymbol{t} \right) W(\mathrm{d}\boldsymbol{s}), & d > 0, \\ \int_{\mathbb{R}^\nu} \left(\int_{\mathbb{R}^\nu} a_\infty(\boldsymbol{t} - \boldsymbol{s}) (\phi(\boldsymbol{t}) - \phi(\boldsymbol{s})) \mathrm{d}\boldsymbol{t} \right) W(\mathrm{d}\boldsymbol{s}), & d < 0, \\ \sigma \int_{\mathbb{R}^\nu} \phi(\boldsymbol{s}) W(\mathrm{d}\boldsymbol{s}), & d = 0, \end{cases}$$

where $W(\mathrm{d}\boldsymbol{s})$ is a real-valued Gaussian random measure with mean zero and variance $\mathrm{d}\boldsymbol{s}$ and

$$a_\infty(\boldsymbol{t}) = |\boldsymbol{t}|^{2d-\nu} \ell(\boldsymbol{t}/|\boldsymbol{t}|), \quad \boldsymbol{t} \neq \boldsymbol{0}, \qquad d \neq 0,$$

$$\sigma^2 = \Big(\sum_{\boldsymbol{t} \in \mathbb{Z}^\nu} a(\boldsymbol{t}) \Big)^2 = \sum_{\boldsymbol{t} \in \mathbb{Z}^\nu} \mathrm{Cov}(X(\boldsymbol{0}), X(\boldsymbol{t})) < \infty, \qquad d = 0.$$

Proposition (P., Surgailis 2024+)

Let X be as in (3), where A(d) holds for $|d| < \nu/4$. Then

$$\lambda^{-(\nu+4d)/2} \int_{\mathbb{R}^\nu} X([\boldsymbol{t}])\phi(\boldsymbol{t}/\lambda)\mathrm{d}\boldsymbol{t} \overset{\mathrm{d}}{\to} W(\phi), \quad \phi \in L^1(\mathbb{R}^\nu) \cap L^\infty(\mathbb{R}^\nu),$$

where if $d < 0$ then ϕ in addition satisfies

$$\int_{\mathbb{R}^\nu} \Big(\int_{\mathbb{R}^\nu} (\phi(\boldsymbol{t}+\boldsymbol{s}) - \phi(\boldsymbol{s}))^2 \mathrm{d}\boldsymbol{s} \Big)^{1/2} |\boldsymbol{t}|^{2d-\nu} \mathrm{d}\boldsymbol{t} < \infty.$$

- Proof uses as $\lambda \to \infty$ the asymptotics of

$$\mathrm{Var}\Big(\int_{\mathbb{R}^\nu} X([\boldsymbol{t}])\phi(\boldsymbol{t}/\lambda)\mathrm{d}\boldsymbol{t} \Big)$$

- Lahiri, Robinson 2016 and references therein

Properties by Dobrushin 1979

Let $\mathcal{S}(\mathbb{R}^\nu)$ be the Schwartz space of $\phi : \mathbb{R}^\nu \to \mathbb{R}$. Then $\{W(\phi),\, \phi \in \mathcal{S}(\mathbb{R}^\nu)\}$ is

- stationary: if for all $\boldsymbol{a} \in \mathbb{R}^\nu$,

$$W(\phi) \overset{\mathsf{d}}{=} W(\phi(\cdot + \boldsymbol{a})), \quad \phi \in \mathcal{S}(\mathbb{R}^\nu),$$

- self-similar with index $H = (\nu - 4d)/2 \in (0, \nu)$: if for all $\lambda > 0$,

$$W(\phi) \overset{\mathsf{d}}{=} \lambda^{H-\nu} W(\phi(\cdot/\lambda)), \quad \phi \in \mathcal{S}(\mathbb{R}^\nu).$$

Possible extensions:

- anisotropic scaling
- infinite variance
- fractional integration on $\mathbb{R}^\nu$ or graph G

Thank you for your attention