ON A GENERAL KAC-RICE FORMULA FOR THE MEASURE OF A LEVEL SET
GeoStoc24-Tours

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Introduction

Weak Bulinskaya condition

$D = d$

$D > d$

Introduction

The history of the Kac-Rice formula dates back to 1943. Two seminal articles established it for the first time.

- M. Kac. On the average number of real roots of a random algebraic equation.Bull. Amer. Math. Soc.49(1943), 314–320.
- Rice, S. O. Mathematical analysis of random noise.Bell System Tech. J.24(1945), 46–156.

Despite the gap between the two publications, several generalizations are explored in the Rice article, as is evident in Kac's own review.

Mathematical analysis of random noise.
Bell System Tech. J. **24** (1945), 46–156.
This is the second half of a paper whose first half was reviewed previously [same J. **23,** 282–332 (1944); MR0010932]. Starting with the representation

$$I(t) = T^{-\frac{1}{2}} \sum_{k=1}^{N} B(\omega_k)(X_k \cos \omega_k t + Y_k \sin \omega_k t), \quad \omega_k = 2\pi k/T,$$

for the noise current, the author calculates the average number per second of zeros of $I(t)$ and of maxima and minima. For large N and T the average number of zeros is given by the simple formula

$$\pi^{-1}\{-\psi''(0)/\psi(0)\}^{\frac{1}{2}},$$

where

$$\psi(t) = \int_0^\infty B^2(f) \cos 2\pi f t df$$

is the "correlation function" of $I(t)$, and $\psi''(0)$ is assumed to exist. The author also calculates the probability that a zero of $I(t)$ should lie in $(t_1, t_1 + dt_1)$ and another zero in $(t_2, t_2 + dt_2)$. [All these results can also be derived using the methods introduced by the reviewer [Bull. Amer. Math. Soc. **49,** 314–320, 938 (1943); Amer. J. Math. **65,** 609–615 (1943); MR0007812; MR0009655; MR0009061].] Agreement between these theoretical results and experimental data obtained by M. E. Campbell is discussed. A hitherto unsolved problem of great practical importance is to determine the probability that no zero of $I(t)$ should fall within a prescribed interval. The author gives an expression for this probability in the form of an infinite series but this is inadequate for computational purposes. Several sections are devoted to the investigation of analogous problems for the so-called "envelope" of $I(t)$ which is obtained experimentally by passing the noise current through a rectifier.

Both results are based on the variable change formula known as the Area Formula. Which is expressed in this case as follows. Let X be a random process with trajectories in $\mathbf{C}^1(\mathbb{R})$, if f is a a.s. continuous function, then

$$\int_{\mathbb{R}} f(u) N^X_{[a,b]}(u) du = \int_a^b f(X(s))|X'(s)| ds,$$

where

$$N^X_A(u) = \#\{s \in A : \ X(s) = u\}.$$

Giving rise to the Kac counter.

$$N^X_{[a,b]}(u) = \lim_{\delta \to 0} \frac{1}{2\delta} \int_a^b \mathbf{1}_{[u-\delta, u+\delta]}(X(s))|X'(s)|ds \quad a.s.$$

and by Fatou's lemma, and denoting $p_{X(s),X'(s)}(x,x')$ the density function of the vector $(X(s)), X'(s))$, we have

$$\mathbb{E}[N^X_{[a,b]}(u)] \leq \int_a^b [\int_{\mathbb{R}} |x'| p_{X(s),X'(s)}(x,x') dx'] ds.$$

Under Bulinskaya condition

$$\mathbb{P}\{\exists t \in U,\, X(t) = u,\, X'(t) = 0\} = 0,$$

the reverse inequality is also true and so the Kac-Rice formula arises

$$\mathbb{E}[N^X_{[a,b]}(u)] = \int_a^b [\int_{\mathbb{R}} |x'| p_{X(s),X'(s)}(x,x') dx'] ds.$$

Our conference is mainly concerned with the multidimensional extension of this famous formula.

The area formula has two multidimensional generalizations. Its discovery and popularization are the work of Federer's monumental book "Geometric measure theory".

- Area formula $f : \mathbb{R}^D \to \mathbb{R}$ continuous a.s. and if $B \subset T$ is a relatively compact open set and $X : T \subset \mathbb{R}^D \to \mathbb{R}^D$ and $X \in \mathbf{C}^1(T)$ then

$$\int_{\mathbb{R}^D} f(u) N_T^X(u) du = \int_T f(X(s)) |\det X'(s)| ds.$$

- The Coarea formula: $f : \mathbb{R}^d \to \mathbb{R}$ continuous a.s.
  $X : T \subset \mathbb{R}^D \to \mathbb{R}^d$ with $D > d$

$$\int_{\mathbb{R}^d} f(u)\sigma_{D-d}(\mathcal{L}_u(B))du = \int_B f(X(s))\Delta(s)ds.$$

Here $\Delta(s) = (\det(X'(s)(X'(s))^T))^{\frac{1}{2}}$, σ_{D-d} is the $D-d$-dimensional Hausdorff measure on $\mathbb{R}^D$ and

$$\mathcal{L}_u(B) = \{t \in B : X(t) = u\},$$

is the level set.

Our main concerns is the regularity of this level set for $\mathbf{C}^1$ random fields. Under further smoothness conditions on the paths of X the Morse-Sard type arguments and implicit function theorem imply that for a.s. level $u \in \mathbb{R}^d$ the level set is a $D - d$ manifold then in the coarea formula we can replace the Hausdorff measure a by the Lebesgue measure.
But in applications we need a Kac-Rice formula for all u.

If we assume $\mathbf{C}^1$ regularity and the strong Bulinskaya theorem

$$\mathbb{P}\{\exists t \in U, X(t) = u, \Delta(t) = 0\} = 0,$$

the set $\mathcal{L}_u$ is a.s. a manifold.
Satisfying this type of condition needs to demand increasing smoothness of the X-paths as the dimension D increases.

Our contribution is to introduce instead the weak Bulinskaya condition

$$\sigma_{D-d}(\{t \in T : X(t) = u, \Delta(t) = 0\}) = 0.$$

Note that σ_0 coincides with the counting measure. Then if $D = d$ the two conditions are the same.

We have the following key proposition

Proposition 1. *Let* $T \subset \mathbb{R}^D$ *be an open set and let* $X : T \to \mathbb{R}^d$ *be a* C^1 *random field. Let* $u \in \mathbb{R}^d$. *Assume that the density* $p_{X(t)}$ *of* $X(t)$ *satisfies*

$$p_{X(t)}(v) \leq C \text{ for all } t \in T \text{ and } v \text{ in some neighbourhood } V_u \text{ of } u. \quad (1)$$

Then

$$\sigma_{D-d}\left(\{t \in T : \; X(t) = u, \, \Delta(t) = 0\}\right) = 0, \; a.s..$$

Some comments are in order:

1. The proof of this proposition, uses local time and extends existing results considerably. (Adler-Taylor [2, Lemma 11.2.11] and Azaïs-Wschebor [7, Proposition 1.20, Proposition 6.11], [26, Lemma 6].)
2. When $D > d$, since the regular part of the level set is a $\mathbf{C}^1$ manifold of dimension $D - d$, it implies that the level set is a.s. an $(D - d)$-rectifiable set. In particular its $(D - d)$-dimensional Hausdorff measure coincides with the associated geometric measure of the regular part of $\mathcal{L}_u$, i.e. the Riemaniann measure induced by the inherited Riemannian structure from R^D.

We will give some ideas about the proof. Let us begin with the Kac-Rice formula for a.s. u for every Borel set $B \subset T$, we have

$$\mathbb{E}(\sigma_{D-d}(\mathcal{L}_u(B))) = \int_B \mathbb{E}\big(\Delta(t)|X(t) = u\big)\, p_{X(t)}(u)\; dt \qquad (2)$$

for almost every $u \in \mathbb{R}^d$. The proof is easy

Let $g : \mathbb{R}^d \to \mathbb{R}$ be a test function, a bounded Borel non-negative function. By the co-area formula one gets

$$\int_{\mathbb{R}^d} g(u)\sigma_{D-d}\left(X^{-1}(u) \cap B\right) du = \int_B g(X(t))\,|\Delta(t)|\,dt.$$

Take expectations on both sides. $\mathbb{E}\big(|\Delta(t)|/X(t) = u\big)$ is well defined for almost every u.

Applying Fubini's theorem and conditioning by $X(t) = u$, we get

$$\int_{\mathbb{R}} g(u)\mathbb{E}\big(\sigma_{D-d}(\mathcal{L}_u(B))\big)du$$
$$= \int_{\mathbb{R}} g(u)du \int_{B} \mathbb{E}\big(|\Delta(t)||X(t) = u\big)p_{X(t)}(u)dt.$$

By duality, since $g(\cdot)$ is arbitrary, the terms in factor of $g(u)$ on both sides are equal for almost every u, giving the result.

The Hausdorff measure.

For a given $K \subset \mathbb{R}^D$, recall that $\sigma_\ell(K)$ is the ℓ-dimensional Hausdorff measure given by $\sigma_\ell(K) = \lim_{\varepsilon \to 0} \sigma_\ell^\varepsilon(K)$, where σ_ℓ^ε is the ℓ-dimensional Hausdorff pre-measure on $\mathbb{R}^D$ given by

$$\sigma_\ell^\varepsilon(K) := \alpha_\ell \inf\{\sum_i {\rho_i}^\ell, \text{ for a covering } \{B_D(x_i, \rho_i)\} \text{ of } K, \ \rho_i < \varepsilon\},$$

where $\alpha_\ell := \lambda_\ell(B_\ell(0,1))$ is the Lebesgue measure of the unit ball in $\mathbb{R}^\ell$. We need a technical lemma.

Lema

Let K be a compact set of $\mathbb{R}^D$, and assume $d < D$. Suppose $\sigma_{D-d}(K)$ is finite. Then there exist constants C_1 and C_2, depending on d and D, such that, for ε sufficiently small:

- *There exists an "ε-packing" (a collection of disjoint balls with center belonging to K and with radius ε) with cardinality $I(\varepsilon)$,*

$$I(\varepsilon) \geq C_1 \varepsilon^{d-D} \sigma_{D-d}(K).$$

- *The parallel set $K^{+\varepsilon} = \bigcup_{x \in K} B_D(x, \varepsilon)$ satisfies*

$$\sigma_D(K^{+\varepsilon}) \geq C_2 \varepsilon^d \sigma_{D-d}(K).$$

Proof of the proposition.

We assume that $u = 0$ and that the considered set T is compact. Define the density of the occupation measure LT, the *local time*, as

$$\mathrm{LT} := \liminf_{\delta \to 0} \mathrm{LT}(\delta)$$

$$:= \liminf_{\delta \to 0} \frac{1}{\lambda_d(B_d(0,\delta))} \sigma_D\left(\{t \in T : \|X(t)\| \leq \delta\}\right) ,$$

where $\lambda_d(B_d(0,\delta)) = \alpha_d\, \delta^d$ is the volume of the ball $B_d(0,\delta)$ with radius δ on $\mathbb{R}^d$. The definition of the local time is in a generalized sense. Fubini and the hypothesis for the density implies

$$\mathbb{E}[\mathrm{LT}] \leq C\sigma_D(T),$$

thus LT is a random variable finite a.s.

Let $M := \max_{t \in T} \lambda_{max}(X'(t))$, where λ_{max} is the greatest singular value. Let

$$N(\varepsilon) := \sup_{t \in T;\, \|v\| < \varepsilon} \frac{\|X(t+v) - X(t) - X'(t)v\|}{\|v\|}.$$

Then, by compactness, $N(\varepsilon)$ and M are almost surely finite non-negative random variables and $N(\varepsilon) \to 0$, as $\varepsilon \to 0$.

Let $\tilde{\mathcal{L}}_0$ the irregular part of the level set $\mathcal{L}_0$, i.e.,

$$\tilde{\mathcal{L}}_0 := \{t \in T : X(t) = 0, \Delta(t) = 0\}.$$

Let $t \in \tilde{\mathcal{L}}_0$, then $X(t) = 0$, $\mathrm{rk}(X'(t)) = k$ for some $k \in \{0, 1, \dots, d-1\}$.
Let $v_1 \in V_1 := \ker(X'(t))$, where V_1 has dimension $D - k$.
Let $v_2 \in V_2 := V_1^{\perp}$, where V_2 has dimension k.

Thus

$$\begin{aligned}\|X(t+v_1+v_2)\| &\le \|X(t+v_1+v_2) - X(t+v_1)\| + \|X(t+v_1)\| \\ &\le M\|v_2\| + \|v_1\| \cdot N(\|v_1\|).\end{aligned}$$

Suppose ε is such that $N(\varepsilon) < 1$, choose v_1 and v_2 such that

$$\|v_1\| \le \frac{1}{\sqrt{2}}\varepsilon, \quad \|v_2\| \le \frac{1}{\sqrt{2}}\varepsilon \cdot N(\varepsilon). \tag{3}$$

Then we have

$$\left\| X\left(t + \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}\right)\right\| \le \frac{1}{\sqrt{2}}(M+1)\varepsilon \cdot N(\varepsilon).$$

The conditions given on v_1 and v_2 defines a polydisk with volume

$$\text{const}\,\varepsilon^{D-k}\big(\varepsilon \cdot N(\varepsilon)\big)^k,$$

where the constant depends on D, d, and k. The polydisk is included in $\overline{\mathcal{L}_0}^{+\varepsilon}$ and thus in T for sufficiently small ε.

Let us define the event $Z_{D,d} = \{\omega : \sigma_{D-d}(\tilde{\mathcal{L}}_0) > 0\}$, and assume by contradiction that $\mathbb{P}(Z_{D,d}) > 0$.
In the case $D = d$, given $\omega \in Z_{d,d}$, let $\delta = \delta_{\omega,\varepsilon} = \big((M+1)\varepsilon \cdot N(\varepsilon)\big)$. Then the approximated local time $\mathrm{LT}(\delta)$ is greater than

$$\frac{1}{(\varepsilon \cdot N(\varepsilon))^d} \text{ (const) } \varepsilon^{d-k}\big(\varepsilon \cdot N(\varepsilon)\big)^k = \text{ (const) } (N(\varepsilon))^{k-d},$$

which tends to infinity as $\varepsilon \to 0$, giving the contradiction.
The proof for $D > d$ is similar.

Theorem 1:[Rice formula for the expectation] Let $X : T \to \mathbb{R}^d$ be a random field, T an open subset of $\mathbb{R}^D$, $(D \geq d)$.
Assume that:

(i) The sample paths of $X(\cdot)$ are a.s. C^1,

(ii) for each $t \in T$, $X(t)$ admits a continuous density $p_{X(t)}(v)$, which is bounded uniformly in $v \in R^d$ and t in any compact subset of T.

(iii) For **every** $v \in \mathbb{R}^d$, for every $t \in T$, the distribution of $\{X(s), s \in T\}$ conditional to $X(t) = v$ is well defined as a probability and is continuous, as a function of v, with respect to the C^1 topology.

Then, for every Borel set B contained in T and for every level u, one has

$$\mathbb{E}\left(\sigma_{D-d}\left(\mathcal{L}_u(B)\right)\right) = \int_B \mathbb{E}\left(\Delta(t)|X(t) = u\right) p_{X(t)}(u)dt. \qquad (4)$$

The previous theorem implies the following result for the Gaussian case. The most important aspect is that there are almost no conditions.

Theorem 2: Let $X : T \to \mathbb{R}^d$ be a Gaussian random field, T an open subset of $\mathbb{R}^D$, $(D \geq d)$, and $u \in \mathbb{R}^d$, satisfying the following:

(i) The sample paths of $X(\cdot)$ are a.s. C^1;

(ii) for each $t \in T$, $X(t)$ has a positive definite variance-covariance matrix.

Then, (4) holds true. In addition, if B is compact, both sides of (4) are finite and, consequently, the measure $B \mapsto \mathbb{E}\left(\sigma_{D-d}\left(\mathcal{L}_u(B)\right)\right)$ is Radon.

It is impossible in such a short time to give a complete proof of the two previous theorems. We will settle for a few tips to show where the innovation lies.

- First of all, we must say that the Gaussian case is derived from the general one by defining conditional distributions by means of regression formulas.
- With respect to the conditions of Theorem 1. We must say that the introduction of the C^1 topology allows us to obtain much generality and we get the idea from a recent result of Angst and Poly for dimension one.

- The following result, when $D = d$, allows us to approximate the number of roots of the field $X(\cdot) - u$ that we denote by $N_u(X, T)$. If $X_n \to X$ in the $C^1(T, \mathbb{R}^d)$ topology and there are not roots on the boundary of T and the Bulinskaya condition holds. Then $N_u(X_n, T) \to N(X, T)$ a.s.

We turn now to the main result whenever $D = d$. Mainly, three problems may arise:

1. The roots $X(t) = u$ can be associated with a Jacobian $X'(t)$, which is almost singular or even singular, namely, $\Delta(t) = 0$.
2. There can be some roots on the boundary of the considered set.
3. The quantities considered may be too large, causing non-integrability.

All these problems are overcome by a **bounding and tapering argument** followed by a monotone convergence argument. The tapering is here necessary to keep the continuity, which is a key argument. Moreover, convergence in the topology of C^1 allows the approximation by crossings of processes that satisfy the conditions that avoid the problems pointed out before.

Proof $D > d$:

In this case our proof method is different from the usual one. The method will allow us to eliminate the need to demand a high regularity for X when D is large. Let us to point outs that the weak Bulinskaya condition implies that the level set is the union of the regular points $\Delta(t) \neq 0$ with the set of irregular points $\Delta(t) = 0$. That is, the union of a manifold with a set of zero measure. In other words, the set is rectifiable. Our fudamnetal tool is Crofton's formula.

Suppose that we are in the Euclidean space $\mathbb{R}^D$. Given $\ell < D$, we consider the Grassmannian manifold $\mathbb{G}_{D,\ell}$ of ℓ-dimensional subspaces of $\mathbb{R}^D$. Let $d\mathbb{G}_{D,\ell}$ be the Haar measure on this space. Let B be a Borel set in $\mathbb{R}^D$. We define the *m-integral geometric measure* of B by

$$\mathcal{I}_{D,m}(B) := c_{D,m} \int_{V \in \mathbb{G}_{D;D-m}} d\mathbb{G}_{D,D-m}(V) \int_{y \in V^{\perp}} d\lambda_m(y) \, \# \left\{ B \cap \ell_{V,y} \right\},$$

where $\ell_{V,y}$ is the affine linear space $\{y + V\}$, and where we naturally identify the element of the grassmanian $V \in \mathbb{G}_{D,D-m}$ with the associated subspace of codimension m on $\mathbb{R}^D$.

An important property of working with this formula is that, if B is a rectifiable set we have

$$\sigma_d(B) = \mathcal{I}_{D,m}(B).$$

A simple way to characterise m-rectifiable sets is to describe then as the union of countably many C^1 manifolds and a set of σ_m measure zero. This fact makes possible to apply this formula for obtaining the Kac-Rice formula by using our weak Bulinskays's condition.

By Fubini's theorem

$$\mathbb{E}\left(\sigma_{D-d}(\mathcal{L}_u(B))\right)$$
$$= c_{D,D-d} \int_{V \in \mathbb{G}_{D,d}} d\mathbb{G}_{D,d} \int_{y \in V^{\perp}} d\lambda_{D-d}(y)\, \mathbb{E}\#\left\{B \cap \ell_{V,y}\right\},$$

where

$$\mathbb{E}\#\left\{B \cap \ell_{V,y}\right\} = \mathbb{E}\Big(N_u\big(X, B \cap (V + y)\big)\Big).$$

Applying the Kac-Rice formula, for $D = d$, and again Fubini's theorem:

$$\mathbb{E}\left(\sigma_{D-d}(\mathcal{L}_u(B))\right)$$
$$= c_{D,D-d} \int_{V \in \mathbb{G}D,d} d\mathbb{G}_{D,d}(V) \int_B \mathbb{E}\big(|\det(X'(t))(\pi_V)^\top|/X(t) = u\big)$$
$$\times\, p_{X(t)}(u)dt,$$

After a new inversion of integral we get the result identity

$$c_{D,D-d} \int_{V \in G(D,d)} |\det(X'(t)(\pi_V)^\top)| d\mathbb{G}_{D,d}(V) = \Delta(t),$$

elle vit apparaître le matin et se tut discrètement...
Thank you.